International Journal of Engineering Technology & Advanced Learning
ISSN: 3107-4227, Vol. 01 (01), March 2025, pp. 41-47

Effect of prey refuge on a delayed prey-predator model in periodic environment

Suhaib Hamid[1], Poonam Sinha[2], Aadil Hamid[3]

[1]*Department of Mathematics, University of Kashmir., Jammu Kashmir, India*
[2]*Department of Mathematics, Govt. S.M.S, Science College, Gwalior, M.P., India*
[3]*Department of Mathematics, Govt. College for Women Parade Jammu, Jammu Kashmir, India*

Abstract

This study investigates a delayed non-autonomous eco-epidemiological predator-prey model incorporating predator infection and prey refuge within a periodically fluctuating environment. The model considers an incubation delay in infection among predators, along with the impact of healthy predator crowding. Theoretical analysis establishes conditions for the existence, boundedness, and uniform persistence of the system's solutions. Through numerical simulations, we examine how varying incubation delays influence system dynamics, revealing a transition from periodic stability to bifurcation and ultimately chaos. Additionally, the effect of prey refuge is explored, demonstrating that moderate prey refuge stabilizes the system, while excessive refuge disrupts equilibrium and induces chaotic behavior. These findings highlight the complex interplay between time delays, periodic environments, and ecological stability, offering insights into predator-prey interactions in fluctuating ecosystems.

*Keywords***:** *Predator-prey, Incubation delay, Chaos, Periodic environment.*

---

1. Introduction

Mathematical modeling has been instrumental in understanding the complex interactions within biological and ecological systems. Predator-prey models, in particular, have been extensively used to analyze population dynamics, species interactions, and ecological stability [1]. The inclusion of time delays in such models is crucial, as many biological interactions are not instantaneous [2]. In predator-prey dynamics, delays can arise due to factors such as gestation periods, maturation time, and handling time of prey by predators [3].

One significant ecological factor influencing predator-prey interactions is the presence of prey refuge. Prey refuge refers to a strategy where a fraction of the prey population is shielded from predation, either by hiding in physical shelters or by behavioural adaptations [4]. The impact of prey refuge on predator-prey models has been widely studied [5, 6, 7, 8], demonstrating that it can stabilize population oscillations [9] and prevent predator extinction [10, 11]. However, the effect of prey refuge in a delayed environment with periodic fluctuations remains an area that requires further exploration. The role of delays in predator-prey models has been extensively examined within the field of mathematical ecology, with early studies, such as Gopalswamy [12] , exploring how time delays influence the stability and oscillatory behavior of population dynamics. In epidemiological contexts, the impact of incubation delays on disease spread has been analyzed by Diekmann and Heesterbeek [13], offering valuable insights into how such delays affect population dynamics. The concept of prey refuge itself has been studied in numerous predator-prey models. Huang et al. [14] for instance, investigated its impact on the permanence and global stability of delayed Lotka-Volterra systems, revealing that prey refuge plays a crucial role in mitigating predator-prey oscillations and stabilizing populations.

Recent advancements have incorporated periodic environments into predator-prey models, acknowledging the critical influence of seasonal and environmental fluctuations on species interactions [15]. Geritz and Kisdi [16] examined evolutionary dynamics in predator-prey interactions under fluctuating environmental conditions, suggesting that evolutionary adaptations in predator and prey traits can significantly alter population dynamics The impact of fear in a delayed predator-prey system with prey refuge in the presence of additional food was studied by Wang et al [17], highlighting how fear-induced behaviors can influence population stability. Time-delayed predator-prey interactions with the benefit of antipredation responses in the presence of refuge is studied by [18], providing insights into how prey strategies can affect predator-prey dynamics. Ruan and Wei (2001) examined Hopf bifurcation phenomena in delayed ecological and epidemiological models, revealing that delay- induced bifurcations could shift system dynamics from stability to oscillatory regimes. Santra et al. [19] investigated time-delayed predator-prey interactions

E-mail ID: suhaibhamid9@gmail.com

with the benefit of anti-predation responses in the presence of refuge. Abbasi et al. [20] explored multiple periodicities in a predator-prey model with prey refuge, offering a deeper understanding of the complex periodic behaviors that can arise in such systems. These recent contributions underscore the evolving nature of research in this field and the importance of considering various factors, including fear effects, anti-predation responses, and complex periodic behaviors, in predator-prey dynamics. Despite these advancements, the combined effect of prey refuge, time delays, and periodic environments remain underexplored. Our study builds upon previous research by incorporating these factors into a unified mathematical framework. By establishing conditions for existence, uniqueness, and uniform persistence, we provide a comprehensive analysis of how delayed predation and disease incubation influence predator-prey interactions in a fluctuating environment.

2. Model Formulation

The model is described by the following system of nonlinear differential equations:

$$\frac{dx_1}{dt} = r(t)x_1(t) - \alpha_1(t)(1 - m(t)x_3(t))x_1(t)x_3(t) - \eta(t)x_1(t)x_2(t) - d_1(t)x_1(t)$$

$$\frac{dx_2}{dt} = \eta(t)x_1(t-\tau)x_2(t-\tau) - \alpha_2(t)x_2(t)x_3(t) - d_2(t)x_2(t) \qquad (1)$$

$$\frac{dx_3}{dt} = e_1(t)\alpha_1(t)(1 - m(t)x_3(t))x_1(t)x_3(t) + e_2(t)\alpha_2(t)x_2(t)x_3(t) - d_3(t)x_3(t)$$

The present section contains a delayed non-autonomous prey-predator model incorporating Healhty Prey, Infected Prey, and Predator with population densities $x_1(t)$, $x_2(t)$, and $x_3(t)$, respectively. All three populations lie within the considered region for time $t > 0$. The parameter description of the proposed model is: r is the logistic growth rate of the prey species, and α_1 is the predation rate of healthy prey and α_2 is predation rate of infected prey. Here, we consider η as disease transmission rate and τ is the incubation delay. The conversion coefficient of healthy prey is e_1 and that of infected prey is e_2 rate of prey. The parameter m denotes healthy prey refuge coefficient. Finally, d_1 , d_2 and d_3 are the death rates of healthy prey, infected prey and predator. All the earlier assumptions lead to the underlying non-autonomous model.

Theorem 1: The system possesses a unique solution over the interval $[-\tau, \infty)$ for the given initial conditions:

$$x_1(\phi) = \psi_1(\phi), \quad x_2(\phi) = \psi_2(\phi), \quad x_3(\phi) = \psi_3(\phi), \quad \phi \in [-\tau, 0],$$

where $\psi_1(\phi), \psi_2(\phi), \psi_3(\phi)$ are continuous and non-negative initial functions. $j_i(\varphi) \geq 0, \varphi \in [-\tau, 0], i = 1,2,3.$
(2)

3. Preliminaries

The following subsections demonstrate fundamental results such as the permanence, existence, and uniqueness of the solution to model (1).

3.1. Existence and uniqueness of the solution

Theorem 2: The system (1) possesses a unique solution over the interval $[-\tau, \infty)$ for the initial conditions (2).

Proof. We know that the right-hand side of the system (1) is locally Lipschitzian and completely continuous over C, has a unique solution $(x_1(t), x_2(t), x_3(t))$ of model (1) satisfying initial conditions (2) on $[0, \alpha)$ where $0 < \alpha \leq +\infty$. Now, referring to the first equation of model (1), we can observe that:

$$x_1(t) = x_1(0) \;\; exp \int_0^t [r(v) - \alpha(v)(1 - mx_3(v))x_3(v) - \eta(v)x_2(v) - d_1(v)]\, dv > 0 \;\; \forall\, t \geq 0.$$

Also, using model (1)'s third equation, we have

$$x_3(t) = x_3(0) \;\; exp \int_0^t [e_1(v)\alpha(v)(1 - mx_3(v))x_1(v) + e_2(v)\alpha_2(v)x_2(v) - d_3(v)]\, dv > 0 \;\; \forall \;\; t \geq 0.$$

Next, we need to show that $x_2(t) > 0 \;\; \forall \;\; t \geq 0$. Suppose that $x_2(t) \leq 0 \;\; \forall \;\; t \geq 0$, then $\exists$ a $t_1 > 0$ such that $x_2(t_1) = 0$ and $x_2(t) \geq 0 \;\; \forall t \in [-\tau, t_1]$. Moreover,

$$\frac{dx_2(t)}{dt} \geq -\alpha_2(t)x_2(t)x_3(t) - c(t)x_2(t) \qquad \forall t \in [0, t_1] \qquad (2)$$

$$\Rightarrow x_2(t) \geq x_2(0) \quad exp \int_0^t [-\alpha_2(v)x_3(v) -_2(v)]dv > 0 \qquad \forall t \in [0, t_1].$$

This contradicts our initial supposition. Thus, $x_2(t) > 0, \ \forall \quad t \geq 0$.

3.2 Permanence

This subsection explores the enduring nature of the system (1), using the initial conditions (2). We shall demonstrate how the system will exhibit uniform persistence under specific conditions. Moreover, all integrants of system (1) will survive over a long time under certain conditions. For a bounded and continuous function, $p(t)$ is specified on the interval. $[0, +\infty)$, let $p^l = inf_{t \leq 0} p(t)$ and $p^u = sup_{t \geq 0} p(t)$.

Definition 3.1 If a_i and b_i, $i = 1,2,3$, are positive constants that ensure:

$$a_1 \leq lim_{t\to\infty} inf \ x_1(t) \leq lim_{t\to\infty} \ sup \ sup \ \ x_1 \ (t) \leq b_1,$$
$$a_2 \leq lim_{t\to\infty} inf \ x_2(t) \leq lim_{t\to\infty} sup \ x_2(t) \leq b_2,$$
$$a_3 \leq lim_{t\to\infty} inf x_3(t) \leq lim_{t\to\infty} sup x_3(t) \leq b_3,$$

holds for each solution $(x_1(t), x_2(t), x_3(t))$ of model (1) satisfying initial conditions (2), then model described by (1) is considered uniformly persistent.

Let us consider the system [21],

$$\dot{y} = py(t-\tau) - qy(t) - sy^2(t),$$

where $p, q, s, \tau > 0$; $y(t) > 0$ for $-\tau \leq t \leq 0$. Then, we have

$$\lim_{t\to\infty} y(t) = \begin{cases} \frac{p-q}{s} & p > q \\ 0 & p < q \end{cases}.$$

Theorem 3: Assume that $X(t) = (x_1(t), x_2(t), x_3(t))$ represent solution of the model (1) satisfying initial conditions (2). Consider that model (1) adheres to the following conditions:

$$\frac{\eta_2^l}{(\gamma_1-\lambda)^u} < K^* = max\left\{\left(\frac{r^u}{d_1^l}\right), \left(\frac{r^u}{d_1^l}\right)\left(\frac{\eta^u}{d_2^l}\right)\right\} \tag{3}$$

then $\exists$ a $\Upsilon_3 > 0$ so that

$$x_1(t), x_2(t) < K_3, x_3(t) \leq K_4 \quad \forall t \geq \Upsilon_3, \tag{4}$$

where $K_3 > K^*$ and $K_4 > \frac{(\gamma_1-\lambda)^u K^* - \eta_2^l}{d_2^l}$.

Proo: Let $K_1 > \left(\frac{r^u}{d_1^l}\right)$. Considering the system (1)'s first equation, we have

$$\dot{x}_1(t) \leq x_1(t)[r(t) - d_1(t)] \leq x_1(t)[r^u - d_1^l].$$

Thus, if $x_1(0) \leq K_1$, then $x_1(t) \leq K_1, \quad \forall t \geq 0$. If $S(0) > K_1$ and let $-\beta_1 = K_1(r^u - d_1^l), \beta_1 > 0$, then $\exists$ an $\in > 0$,

such that if $t \in [0, \in), x_1(t) > K_1$ and we have $\dot{x}_1(t) < -\beta_1 < 0$. Therefore, $\exists$ a $\Upsilon_1 > 0$ and $\Upsilon_3 > 0$ such that

$$x_1(t) \leq K_1 \text{ and } x_2(t) \leq K_4 \quad \forall t \geq \Upsilon_1, \Upsilon_3.$$

Using system (2)'s third equation, we get

$$\dot{x}_2(t) \leq \eta^u(t) K_1 x_2(t-\tau) - d_2^l x_2(t), \ \ t \geq \Upsilon_1 + \tau. \tag{5}$$

We deduce from Lemma (4.2) that $\exists$ a $\Upsilon_2 \geq \Upsilon_1 + \tau$ so that $x_2(t) \leq K_2, \ \ \forall t \geq \Upsilon_2$, where $K_2 > \frac{\eta^u K_1}{d_2}$

It is feasible to select K_1 sufficiently in proximity to $\left(\frac{r^u}{d_1^l}\right)$. Hence, $x_1(t), x_2(t) \leq K_3$, where

$$K_3 > K^* = max\left\{\left(\frac{r^u}{d_1^l}\right), \left(\frac{r^u}{d_1^l}\right)\left(\frac{\eta}{u} d_2^l\right)\right\}, \quad \forall t \geq \Upsilon_2.$$

Using system (2)'s third equation, we obtain

$$\dot{x}_3(t) \leq x_3(t)[(e_1\alpha_1)^u(1-m)K_3 + (e_2\alpha_2)^l - d_3^l], \ \ \forall t \geq \Upsilon_3.$$

Therefore, $\exists$ a $\Upsilon_3 > 0$ so that $x_3(t) \leq K_4 \quad \forall t \geq \Upsilon_3$, where $K_4 > \frac{((e_1\alpha_1)^u(1-m)K_3+(e_2\alpha_2)^l}{d_3^l}$, since K_3 can be chosen sufficiently close to K^*.

Theorem 4: The system (1), considering initial condition (2), exhibits uniform persistence when the subsequent condition is satisfied:

$$\frac{\eta_2^l}{(\gamma_1-\lambda)^2} < K^* = max\left\{\left(\frac{b^u}{d_1^l}\right), \left(\frac{b^u}{d_1^l}\right)\left(\frac{\gamma_2^u}{\eta_3^l}\right)\right\} \quad < min\left\{\frac{d_3^l r^l + \alpha_1^u e_2^l - \frac{(d_1^u + d_3^u e_2^u)}{e_1^l \alpha_1^l (1+m)^l}}{(d_2+\eta)^u d_3^l + \alpha_1^u e_1^u \alpha_1^u (1-m)^u}\right., \tag{7}$$

Proof. Assume that $X(t) = (x_1(t), x_2(t), x_3(t))$ is a solution of the model (1) satisfying initial conditions (2), with the help of Theorem 4.2, we have

$$\dot{x}_1(t) \geq x_1(t)[(r^l - \alpha_1(1-mM_3)M_3 - \eta^u M_2 - d_1] - d_1)^u K_3 + \alpha_1^u K_4) - d_1^u], \quad \forall t \geq \Upsilon_3.$$

If $x_1(r^l - \alpha_1(1-mM_4)^u M_4 - \eta^u M_3) - d_1 > 0$, then we can choose m_1 such that

$$0 < m_1 < \frac{[(r^l - \alpha_1(1-mM_4)^u M_4 - \eta^u M_3)]}{d_1^u} \Rightarrow (r^l - \alpha_1(1-mM_4)^u M_4 - \eta^u M_3) - d_1^u m_1 > 0.$$

If $x_1(\Upsilon_3) \geq m_1$, then $x_1(t) \geq m_1 \ \forall t \geq \Upsilon_3$. If $x_1(\Upsilon_3) < m_1$, and let $\mu_1 = x_1(\Upsilon_3)[\ \alpha_1(1-mM_4)^u M_4] - \eta^u M_3 > 0$, then $\exists$ an $\in > 0$, such that $x_1(t) < m_1$, and $\dot{x}_1 > \mu_1 > 0, \ \forall t \in [\Upsilon_3, \Upsilon_3 + \in)$. Therefore, $\exists$ a $\Upsilon_4 > \Upsilon_3 > 0$, such that $x_1(t) \geq m_1 \quad \forall t \geq \Upsilon_4$.

Using system (1)'s third equation, we obtain

$$\dot{x}_2(t) \geq \eta x_1(t-\tau)x_2(t-\tau) - \alpha_2 x_2(t)x_3(t) - d_2 x_2(t). \tag{8}$$

$$\geq \eta^l m_1 x_2(t-\tau) - (\alpha_2^u M_3 + d_2)x_2(t) \tag{9}$$

$$\geq \eta^l m_1 x_2(t-\tau) - \in_2 x_2(t) - (\alpha_2^u M_3 + d_2)x_2(t) \quad \forall t \geq \Upsilon_4 + \tau, \tag{10}$$

where $\in_2$ represents any non-negative real number. Suppose that $\eta^l m_1 > \alpha_2^u M_3 + d_2^u$, then by using Theorem (4.2) we may deduce that $\exists$ a $\Upsilon_5 \geq \Upsilon_4 + \tau$ so that $x2_(t) \geq m_2, \ \forall t \geq \Upsilon_5$, where $m_2 < \frac{\eta^l m_1 + \alpha_2^u M_3 + d_2^u}{\in_2}$. Hence,

$$x_1(t), x_2(t) \geq m_3, \ \forall t \geq \Upsilon_5,$$

where $m_3 < m^* = min\{m_1, m_2\}$.

Using system (2)'s third equation, we obtain

$$\dot{x}_3(t) \geq x_3(t)[e_1^l m_1 \alpha_1(1-mx_3) + e_2^u \alpha_2 m_2 - d_3^u].$$

If $(e_1^l m_1 \alpha_1(1-mx_3)) > e_2^u \alpha_2 m_2 - d_3^u$, then we can select m_4 so that $0 < m_4 < \frac{(e_1^l m_1 \alpha_1(1-mx_3)) + e_2^u \alpha_2 m_2 - d_3^u}{d_3^u}$. If $x_3(\Upsilon_5) \geq m_4$, then $x_3(t) \geq m_4, \ \forall t \geq \Upsilon_5$. If $x_3(\Upsilon_5) < m_4$ and suppose $\mu_2 = x_3(\Upsilon_5)[e_1^l m_1 \alpha_1(1-mx_3) + e_2^u \alpha_2 m_2 - d_3^u] > 0$, then $\exists$ an $\in_3 > 0$ such that $x_3(t) < m_4$ and $\dot{x}_3(t) > \mu_2 > 0, \ \forall t \in [\Upsilon_5, \Upsilon_5 + \in_3)$. Thus, $\exists$ a $\Upsilon_6 > \Upsilon_5 > 0$ such that $\Upsilon(t) \geq m_4 \quad \forall t > \Upsilon_6$. After considering the above analysis, we establish that $\exists$ a $\Upsilon_6 > 0$, ensuring that each solution of system (1) satisfying initial conditions (2) eventually enters and remains in the area

$$\Omega = \{(x_1(t), x_2(t), x_3(t)) | m \leq x_1(t) \leq K, m \leq x_2(t) \leq K, m \leq x_3(t) \leq K\}, \qquad \forall t > \Upsilon_6,$$

where $m = min\{m_3, m_4\}$ and $K = max\{K_3, K_4\}$.

4. Numerical simulation

This section numerically simulates various properties of the nonautonomous model (1), like periodic oscillation, bifurcation, chaos, and effect of prey refuge coefficient m. The system's time-dependent parameters vary periodically as a sine wave function.

$$r(t) = r\left(1 + \varrho_1 sin\left(\frac{2\pi t}{365}\right)\right), \alpha_1(t) = \alpha_1\left(1 + \varrho_2 sin\left(\frac{2\pi t}{365}\right)\right), \alpha_2(t) = \alpha_2\left(1 + \varrho_3 sin\left(\frac{2\pi t}{365}\right)\right)$$

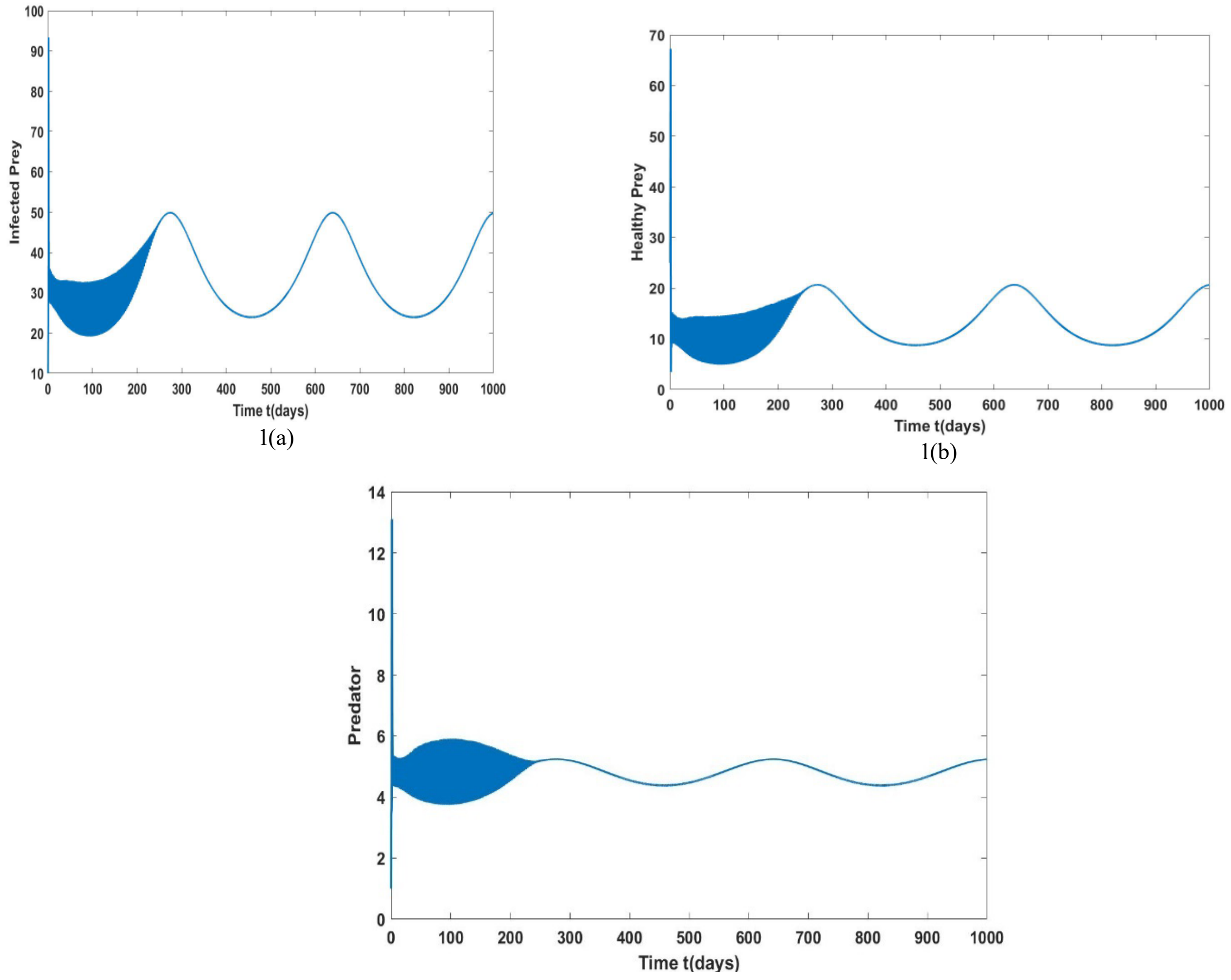
$$\eta(t) = \eta\left(1 + \varrho_4 sin\left(\frac{2\pi t}{365}\right)\right), e_1(t) = e_1\left(1 + \varrho_5 sin\left(\frac{2\pi t}{365}\right)\right), e_2(t) = e_2\left(1 + \varrho_6 sin\left(\frac{2\pi t}{365}\right)\right),$$

$$m(t) = m\left(1 + \varrho_7 sin\left(\frac{2\pi t}{365}\right)\right), d_1(t) = d_1\left(1 + \varrho_8 sin\left(\frac{2\pi t}{365}\right)\right), d_2(t) = d_2\left(1 + \varrho_9 sin\left(\frac{2\pi t}{365}\right)\right),$$

$$d_3(t) = d_3\left(1 + \varrho_{10} sin\left(\frac{2\pi t}{365}\right)\right), \tag{11}$$

$$r = 5.5, \eta = 0.2, \alpha_1 = 0.55, \alpha_2 = 0.4, e_1 = 0.8, e_2 = 0.2, d_1 = 0.29, d_2 = 0.55, d_3 = 0.21, m = 0.1, \varrho_1 = 0.1,$$
$$\varrho_2 = 0.4, \varrho_3 = 0.52, \varrho_4 = 0.052, \varrho_5 = 0.66, \varrho_6 = 0.1, \varrho_7 = 0.1, \varrho_8 = 0.1, \varrho_9 = 0.1, \varrho_{10} = 0.1. \tag{12}$$

The Fig. (1) unveils the simulation results of the nonautonomous system (2) showing the positive periodic oscillations around endemic equilibrium for the parameter set (12) and incubation delay $\tau = 0.46$. While we Increase the incubation delay, the system becomes unstable, crossing the critical value of incubation delay $\tau = 0.564965$ and exhibits bifurcation for $\tau = 0.564965$ showing in the Fig. (2). Finally, Fig. (3) shows the behavior of endemic equilibrium for parameter set (12), incubation delay $\tau = 0.46$, and different values of the prey refuge parameter m. The figure shows that the system shows periodic oscillation for $m = 0.1$, becomes unstable and shows bifurcation for $m = 0.2$, and becomes chaotic for $m = 0.8$.



1(a)

1(b)

Figure 1(c)

Figure 1: Positive periodic solutions of system (1) the parameter set (12) and incubation delay $\tau = 0.46$

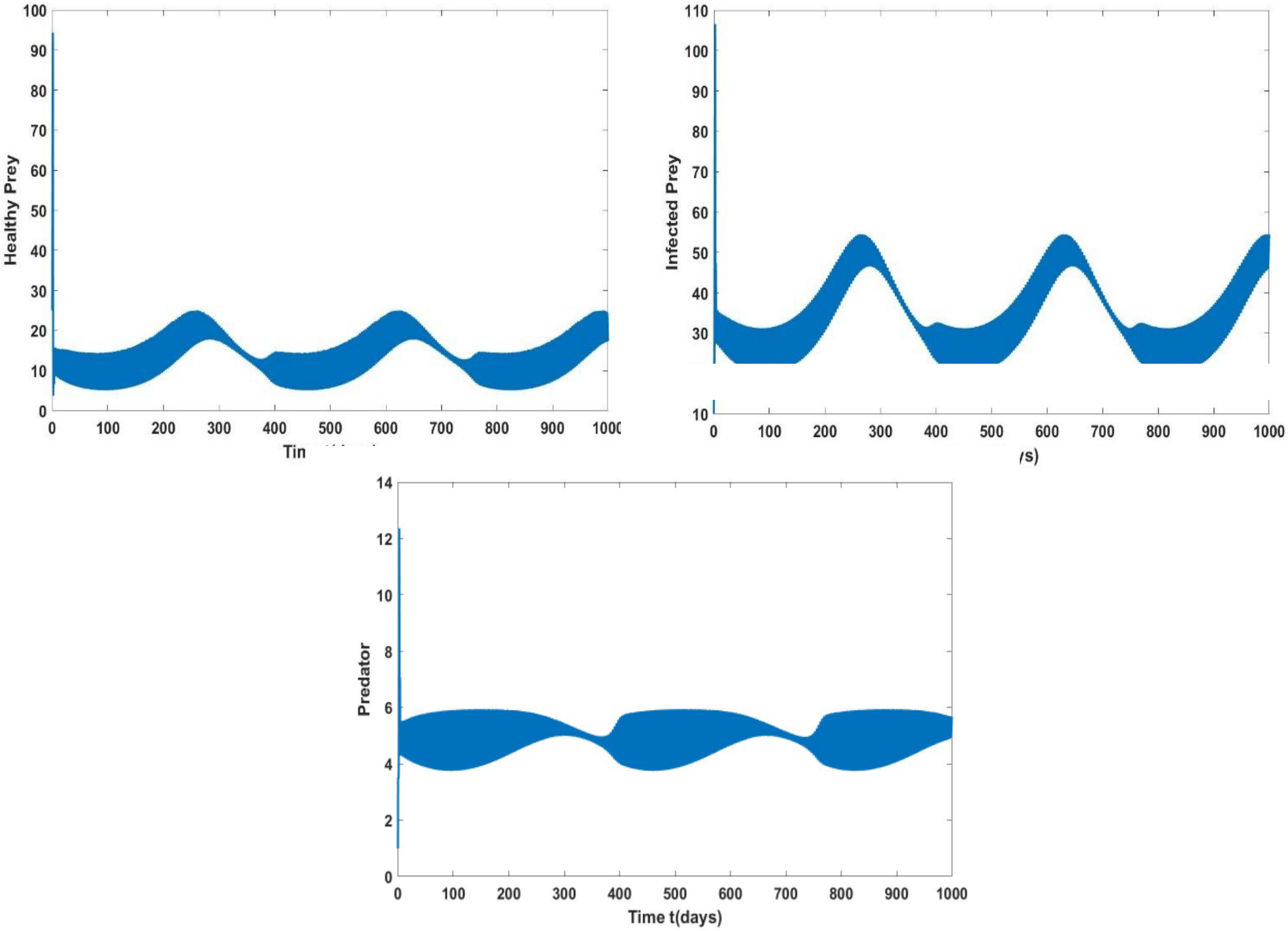
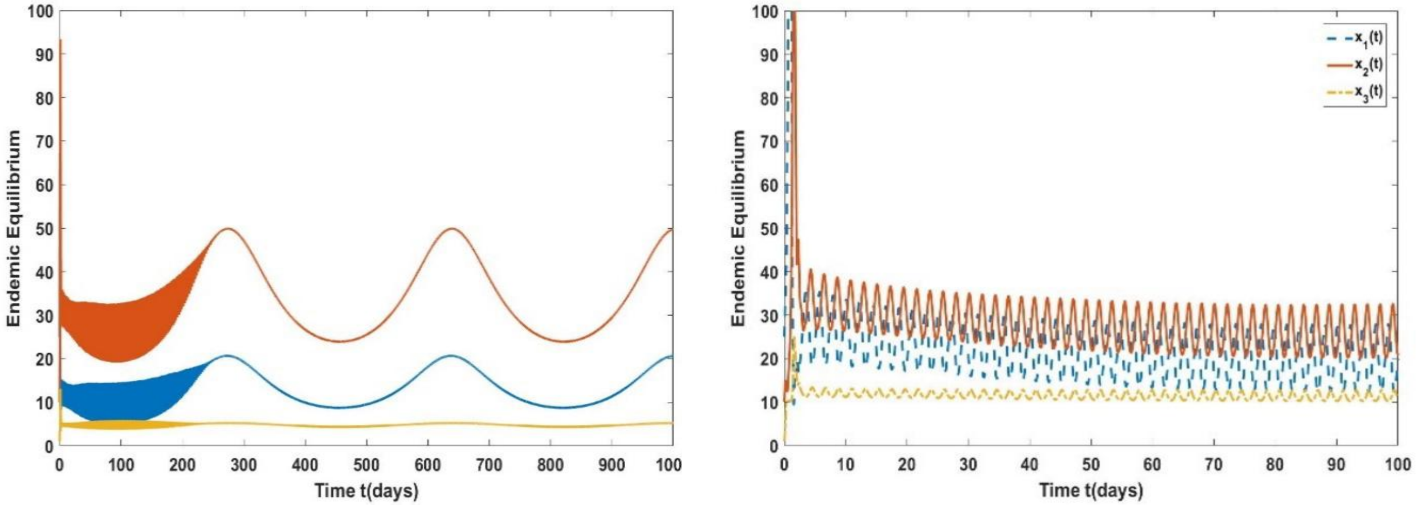


Figure 2: Solution trajectories of the nonautonomous system (2) at $\tau = 0.97$ for the parameter set (12) showing three different showing bifurcation



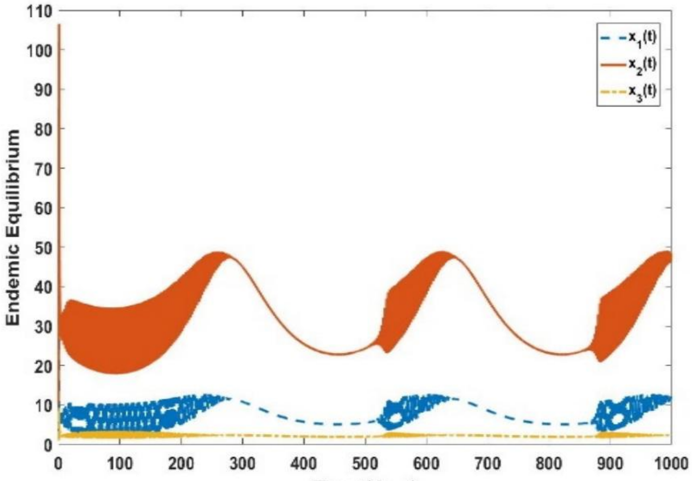


Figure 3: Behavior of the system (1) for $\tau = 0.46$ and for different values of prey refuge coefficients m=0.1, m=0.2, m=0.8 for the parameter set (12).

5. Conclusions

The present research paper proposed a three-compartment prey-predator model with incubation delay of infection in prey and healthy prey refuge. The author analyses the system mathematically and establishes the solution's existence, boundedness, and permanence. Further, the author simulates the system and reveals that the system is periodically stable for the incubation delay $\tau = 0.46$ and shows bifurcation for incubation delay $\tau = 0.97$. Hence, the system is unstable for all diseases with a considerable incubation delay. Again, simulating the system for different values of prey refuge results in the increasing prey refuge parameter; the system loses its stability and leads to chaos. Thus, the system is stable when the prey refuges the predation at a moderate value.

References

[1] J. Maynard-Smith, Models in ecology, Cambridge university press, 1974.117

[2] F. A. Rihan, et al., Delay differential equations and applications to biology, Springer, 2021.118

[3] S. Pandey, A. Sarkar, D. Das, S. Chakraborty, Dynamics of a delay-induced prey–predator system with interaction between immature prey and predators, International Journal of Biomathematics 17 (02) (2024) 2350016.120

[4] J. S. Boon, S. A. Keith, D. A. Exton, R. Field, The role of refuges in biological invasions: A systematic review, Global Ecology and Biogeography 32 (8) (2023) 1244–1271.122

[5] E. Gonz′alez-Olivares, R. Ramos-Jiliberto, Dynamic consequences of prey refuges in a simple model system: more prey, fewer predators and enhanced stability, Ecological modelling 166 (1-2) (2003) 135–146.124

[6] B. Barman, B. Ghosh, Role of time delay and harvesting in some predator–prey communities with different functional responses and intra-species competition, International Journal of Modelling and Simulation 42 (6) (2022) 883–901.126

[7] R. Xu, Global dynamics of a predator–prey model with time delay and stage structure for the prey, Nonlinear Analysis: Real World Applications 12 (4) (2011) 2151–2162.128

[8] M. Banerjee, L. Zhang, Time delay can enhance spatio-temporal chaos in a prey–predator model, Ecological complexity 27 (2016) 17–28.130

[9] W. Chivers, W. Gladstone, R. Herbert, M. Fuller, Predator–prey systems depend on a prey refuge, Journal of theoretical biology 360 (2014) 271–278.132

[10] J. Ghosh, B. Sahoo, S. Poria, Prey-predator dynamics with prey refuge providing additional food to predator, Chaos, Solitons & Fractals 96 (2017) 110–119.134

[11] N. H. Gazi, M. Bandyopadhyay, Effect of time delay on a harvested predator-prey model, Journal of Applied Mathematics and Computing 26 (2008) 263–280.136

[12] K. Gopalsamy, Stability and oscillations in delay differential equations of population dynamics, Vol. 74, Springer Science & Business Media, 2013.138

[13] O. Diekmann, J. A. P. Heesterbeek, Mathematical epidemiology of infectious diseases: model building, analysis and interpretation, Vol. 5, John Wiley & Sons, 2000.140

[14] S. Roy, N. Sk, P. Kumar Tiwari, Bifurcation analysis of autonomous and nonautonomous modified leslie–gower models, Chaos: An Interdisciplinary Journal of Nonlinear Science 34 (2) (2024).142

[15] G. Jiang, Q. Lu, Impulsive state feedback control of a predator–prey model, Journal of Computational and Applied Mathematics 200 (1) (2007) 193–207.

[16] S. A. Geritz, Adaptive dynamics of speciation: Ecological underpinnings stefan ah geritz, ´eva kisdi, g´eza mesz´ena, and johan aj metz, Adaptive speciation (2004) 54.

[17] X. Wang, et al., Impact of fear in a delayed predator-prey system with prey refuge in the presence of additional food, Journal of Mathematical Biology 82 (2021) 1–20.

[18] H. Liu, et al., Multiple periodicities in a predator-prey model with prey refuge, Mathematics 10 (3) (2022) 421

[19] N. Santra, S. Mondal, G. Samanta, Complex dynamics of a predator–prey interaction with fear effect in deterministic and fluctuating environments, Mathematics 10 (20) (2022) 3795.

[20] M. A. Abbasi, Periodic behavior and dynamical analysis of a prey–predator model incorporating the allee effect and fear effect, The European Physical Journal Plus 139 (2) (2024) 113.

[21] X. Song, L. Chen, Optimal harvesting and stability for a two-species competitive system with stage structure, Mathematical biosciences 170 (2) (2001) 173–186.